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ORIGINAL PAPER

Green's functions for non-self-adjoint problems in heat conduction with steady motion

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Abstract Heat conduction in a rectangular parallelepiped that is in steady motion relative to a fluid is studied in this paper. The governing equation consists of the standard heat equation plus lower-order derivative terms with the space variables that represent the effects of the solid flow. The presence of the first-order-derivative terms with the space variables renders the spatial part of the governing differenial equation non-self-adjoint and care must be exercised in defining the new Green's functions to be used in representing the solutions of initial- and boundary-value problems. It is illustrated how the Green's functions may be constructed and how solutions of initial- and boundary-value problems may be obtained that lead to numerical results. Convergence properties of the solutions are also discussed.

Keywords Heat conduction · Non-self-adjoint operators · Green's functions

1 Introduction

In this paper we study heat conduction in a solid in steady motion relative to a fluid. We take the solid to be a rectangular parallelepiped with dimensions $0 < x < L_1, 0 < y < L_2$, and $0 < z < L_3$. The solid is assumed orthotropic and has its principal axes coinciding with the coordinate axes. Initial and boundary conditions, as well as internal heat generation, are considered. The heat-conduction problem to be studied here is based on a model equation which consists of the standard heat equation plus lower-order derivative terms representing the effects of the solid flow. The presence of the lower order derivative terms in the governing equation makes the spatial part of the differential operator non-self-adjoint. The meaning of this "non-self-adjointness" of differential operators is explained in Sect. 2 where a brief review of the relevant operator theory is presented. We shall introduce in Sect. 2 notions of non-self-adjoint operators as they may be relevant to our present work. We shall introduce generalized versions of the Green's functions G and their adjoints G^*, identify the appropriate boundary conditions, show how they may be used in

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the representation of the solutions of the initial- and boundary-value problems and finally, show how to construct the Green's functions so we can deal with this initial-boundary-value heat-conduction problem with solid flow. For background materials on Green's functions for the classical heat equation we refer the readers to Morse and Feshbach [1, Chapter 7]. For background materials on heat conduction in general we refer the readers to Carlslaw and Jaeger [2, Chapters 1–12], Ozisik [3, Chapters 1–6] and Beck et al. [4, Chapters 1–4, 6].

The derivation of the representation for the solutions leads us to consider the questions of appropriate boundary conditions that the Green's functions and their adjoints must satisfy. The appropriateness of the boundary conditions on G and G^* is assured by requiring that the boundary integrals be uniquely determined by the given boundary data for the temperature of the first kind, the second kind or the third kind. Integral representation of solutions in terms of initial and boundary data as well as volumetric energy generation, with the Green's function as the kernel, are given in Sect. 3. Discussions of boundary conditions for the Green's functions are given in Sect. 4.

In Sect. 5 we show how to construct Green's functions by using the large time eigenfunctions of the Green's functions and their adjoints. Section 6 is devoted to two example problems and the related numerical work. Questions on the convergence properties of the Green's function method presented here will also be discussed in Sect. 6. Section 7 contains further discussions and concluding remarks.

2 Formulation of the problem

We consider in this section the formuation of the problem in terms of the Green's function and the initial and boundary data of the problem. The governing equation for heat conduction in a solid in steady motion to a fluid is taken, for example, as in [4]

$$LT(x,y,z,t) \equiv k_x\frac{\partial^2 T}{\partial x^2}+k_y\frac{\partial^2 T}{\partial y^2}+k_z\frac{\partial^2 T}{\partial z^2}-\rho c\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=-\tilde{g}(\vec{x},t),\ t>0, \tag{1}$$

where L stands for the differential operator on the left-hand side of the equation, k_x, k_y, and k_z are the thermal conductivities in the x-, y-, and z-directions, respectively, ρ is density, c is specific heat, $\tilde{g}(\vec{x},t)$ represents the volumetric energy generation, and $\vec{u}=(u,v,w)$ denotes the uniform relative velocity of the moving solid which does not vanish. We allow the thermal conductivities to be unequal so that the solid may be orthotropic.

We have the initial condition for the temperature

$$T(\vec{x};0)=f(x,y,z)=\text{given}, \tag{2}$$

and, on the bounding surfaces x=0 and $x=L_1, y=0$ and $y=L_2$, and z=0 and $z=L_3$ boundary conditions of the first kind (Dirichlet), the second kind (Neumann) or the third kind (Robin) are posed.

Let $\phi(x)$ and $\psi(x)$ be any two smooth differentiable functions of x, $0<x<a$, defined in some function space S, say, the space C_2 of all twice continuously differentiable functions and let L be a linear differential operator defined on C_2. We start with the integral

$$\int_0^a \psi(x)L\phi(x)\mathrm{d}x. \tag{3}$$

We now bring all the differentiations in $L\phi(x)$ to the function ψ by repeated integrations by parts. This results in

$$\int_0^a \psi(x)L\phi(x)\mathrm{d}x=\int_0^a \phi(x)M\psi(x)\mathrm{d}x+\cdots \tag{4}$$

for some linear differential operator M and where "$\cdots$" stands for boundary terms evaluated at the boundaries $x=0$ and at $x=a$. We shall call M the formal adjoint (operator) of L and vice versa. It is clear that if M is the formal adjoint of L, then L is the formal adjoint of M. If the functions ϕ and ψ are further assumed to lie in some subspace of S, say, S_0 of functions that vanish at $x = 0$ and $x = a$ so that all the boundary terms drop out, we then say that L and M are the adjoint of each other. If $L = M$, the operators are said to be self-adjoint.

For a second-order differential operator L given by

$$L = \alpha_0 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta_0 \frac{\mathrm{d}}{\mathrm{d}x} + \gamma_0, \tag{5}$$

where α_0, β_0, and γ_0 are all constant, we have

$$M = \alpha_0 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \beta_0 \frac{\mathrm{d}}{\mathrm{d}x} + \gamma_0. \tag{6}$$

It is seen that L and M can not be self-adjoint unless $\beta_0 = 0$. They are formally self-adjoint if β_0=0, and are self-adjoint if all boundary terms drop out as a result of assumed homogeneous boundary conditions. Self-adjoint operators have many useful and important properties.

We observe that the differential operator L in our present problem is of the second order with constant coefficients. L thus can not be self-adjoint unless the first derivatives terms of T with respect to the spatial variables all drop out, which would imply that $u = v = w = 0$ and there is no solid motion. Thus, the operator L and hence the problem is non-self-adjoint.

Associated with Eq. 1 is the Green's function $G = G(\vec{x}, t; \vec{x}', \tau)$ that satisfies the equation

$$\begin{aligned} LG &\equiv k_x \frac{\partial^2 G}{\partial x^2} + k_y \frac{\partial^2 G}{\partial y^2} + k_z \frac{\partial^2 G}{\partial z^2} - \rho c \left(\frac{\partial G}{\partial t} + u \frac{\partial G}{\partial x} + v \frac{\partial G}{\partial y} + w \frac{\partial G}{\partial z} \right) \\ &= -\rho c \delta(x - x')\delta(y - y')\delta(z - z')\delta(t - \tau), \ t > \tau, \end{aligned} \tag{7}$$

$$G \equiv 0, \ \tau > t. \tag{8}$$

We mention that G gives the thermal effect at the location $\vec{x}$ at time t that is caused by a point impulsive source at $\vec{x}'$ at time τ (Table 1).

The adjoint Green's function $G^*(\vec{x}, t; \vec{x}', \tau)$, of $G(\vec{x}, t; \vec{x}', \tau)$ is defined by

$$G^*(\vec{x}, t; \vec{x}', \tau) = G(\vec{x}, -t; \vec{x}', -\tau) \tag{9}$$

We note that G and G^* both depend on the parameters (u, v, w) which change signs when there is a time reversal. Using (9) and with $\vec{u} \to -\vec{u}$, we can show that G^* satisfies

$$\begin{aligned} L^*G^* &\equiv k_x \frac{\partial^2 G^*}{\partial x^2} + k_y \frac{\partial^2 G^*}{\partial y^2} + k_z \frac{\partial^2 G^*}{\partial z^2} + \rho c \left(\frac{\partial G^*}{\partial t} + u \frac{\partial G^*}{\partial x} + v \frac{\partial G^*}{\partial y} + w \frac{\partial G^*}{\partial z} \right) \\ &= -\rho c \delta(\vec{x} - \vec{x}')\delta(t - \tau)), \ t < \tau, \end{aligned} \tag{10}$$

$$G^* \equiv 0, \ \tau < t, \tag{11}$$

where the operator L^* is defined by the left-hand side of Eq. (10) above. The adjoint Green's function G^* gives the thermal effect at $\vec{x}'$ and τ due to a point impulsive source at $\vec{x}$ and t travelling backwards in time.

In Appendix A we shall, by using Eqs. 7, 8 and 10, 11, prove the so-called "reciprocity property" of G and G^* given as in Eq. 12 or, equivalently, in Eq. 13 below

$$G(\vec{x}, t; \vec{x}', \tau) = G(\vec{x}', -\tau; \vec{x}, -t), \tag{12}$$

$$G(\vec{x}, t; \vec{x}', \tau) = G^*(\vec{x}', \tau; \vec{x}, t). \tag{13}$$

This reciprocity property is useful as it enables us to derive the following equations for G and G^* regarded as functions of (x', y', z', τ)

$$k_x \frac{\partial^2 G}{\partial x'^2} + k_y \frac{\partial^2 G}{\partial y'^2} + k_z \frac{\partial^2 G}{\partial z'^2} + \rho c \left(\frac{\partial G}{\partial \tau} + u \frac{\partial G}{\partial x'} + v \frac{\partial G}{\partial y'} + w \frac{\partial G}{\partial z'} \right) = -\rho c \delta(\vec{x} - \vec{x}')\delta(t - \tau), \ t > \tau, \tag{14}$$

$$G \equiv 0, \ \tau > t, \tag{15}$$

Table 1 Nomenclaure

c	Specific heat
$D = \frac{\rho c u L}{k_x}$,	Dimensionless constant $= \frac{Pe}{2}$
$\tilde{g}(\vec{(x)}, t)$	Volume energy generation
Pe	Peclet number
$F_X,\ H_X,\ F_Y, \ldots$	Spatial eigenfunctions in Green's functions G and G^*
$G_X,\ G_Y,\ G_Z$	One-dimensional Green's functions for the temperature
$G_X^*,\ G_Y^*,\ G_Z^*$	Adjoint Green's functions of G_X, G_Y, and G_Z
$G(x, y, z, t, x', y', z', \tau)$	Three-dimensional Green's function
$k_x,\ k_y,\ k_z$	Thermal conductivities in the x-, y-, and z-directions
q	Heat flux
q_0	Prescribed surface heat flux
T(x,y,z,t)	Temperature
$T_s(x, y, z)$	Steady state temperature
$T_{c.t.}$	Complementary transient temperature
T_0	Prescribed surface temperature
$\vec{(u)} = (u, v, w)$	Solid velocity relative to that of the fluid
R	Eigenvalue parameter in the x-direction
Greek	
α	Thermal diffusivity
κ, λ	Eigenvalue parameters in the x-direction
μ, γ	Eigenvalue parameters in the y-direction
η	Eigenvalue parameter in the z-direction
ρ	Density
σ	$= t - \tau$, cotime

$$k_x \frac{\partial^2 G^*}{\partial x'^2} + k_y \frac{\partial^2 G^*}{\partial y'^2} + k_z \frac{\partial^2 G^*}{\partial z'^2} - \rho c \left(\frac{\partial G^*}{\partial \tau} + u \frac{\partial G^*}{\partial x'} + v \frac{\partial G^*}{\partial y'} + w \frac{\partial G^*}{\partial z'} \right) = -\rho c \delta(\vec{x} - \vec{x}') \delta(t - \tau),\ t < \tau, \tag{16}$$

$$G^* \equiv 0,\ t > \tau. \tag{17}$$

We shall use Eqs. 14 through 17 above to derive integral representations for the solutions in the next section.

3 Representation of solutions

We now derive integral representations for solutions of initial- and boundary-value problems in terms of the Green's functions and the given data. We rewrite Eq. 1 in terms of (x', y', z', τ).

$$k_x \frac{\partial^2 T}{\partial x'^2} + k_y \frac{\partial^2 T}{\partial y'^2} + k_z \frac{\partial^2 T}{\partial z'^2} - \rho c \left(\frac{\partial T}{\partial \tau} + u \frac{\partial T}{\partial x'} + v \frac{\partial T}{\partial y'} + w \frac{\partial T}{\partial z'} \right) = -\tilde{g}(\vec{x}', \tau). \tag{18}$$

We multiply Eq. 14 by $T(x', y', z', \tau)$. We then multiply Eq. 18 by $G(x, y, z, t; x', y', z', \tau)$, here treated as a function of (x', y', z', τ) that satisfies Eq. 14. We subtract these two equations, and integrate the resulting equation with respect to x' from 0 to L_1, with respect to y' from 0 to L_2, with respect to z' from 0 to L_3, and with respect to τ from 0 to t. We have

$$\int_0^t \int_0^{L_3} \int_0^{L_2} \int_0^{L_1} \left(k_x \left(T \frac{\partial^2 G}{\partial x'^2} - G \frac{\partial^2 T}{\partial x'^2} \right) + k_y \left(T \frac{\partial^2 G}{\partial y'^2} - G \frac{\partial^2 T}{\partial y'^2} \right) + k_z \left(T \frac{\partial^2 G}{\partial z'^2} - G \frac{\partial^2 T}{\partial z'^2} \right) \right) \mathrm{d}x' \mathrm{d}y' \mathrm{d}z' \mathrm{d}\tau$$
$$+ \rho c \int_0^t \int_0^{L_3} \int_0^{L_2} \int_0^{L_1} \left(\left(T \frac{\partial G}{\partial \tau} + G \frac{\partial T}{\partial \tau} \right) + u \left(T \frac{\partial G}{\partial x'} + G \frac{\partial T}{\partial x'} \right) + v \left(T \frac{\partial G}{\partial y'} + G \frac{\partial T}{\partial y'} \right) \right.$$

$$+w\left(T\frac{\partial G}{\partial z'}+G\frac{\partial T}{\partial z'}\right)\Bigg)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau=\int_0^t\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}G\tilde{g}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau$$
$$-\rho c\int_0^t\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}T(x',y',z',\tau)\delta(x-x')\delta(y-y')\delta(z-z')\delta(t-\tau)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau. \tag{19}$$

It follows that we have

$$\rho cT(x,y,z,t)=-\int_0^t\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}\left(k_x\left(T\frac{\partial^2G}{\partial x'^2}-G\frac{\partial^2T}{\partial x'^2}\right)+k_y\left(T\frac{\partial^2G}{\partial y'^2}-G\frac{\partial^2T}{\partial y'^2}\right)\right.$$
$$\left.+k_z\left(T\frac{\partial^2G}{\partial z'^2}-G\frac{\partial^2T}{\partial z'^2}\right)\right)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau$$
$$-\rho c\int_0^t\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}\left(\frac{\partial}{\partial\tau}(GT)+u\frac{\partial}{\partial x'}(GT)+v\frac{\partial}{\partial y'}(GT)+w\frac{\partial}{\partial z'}(GT)\right)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau$$
$$+\int_0^t\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}G\tilde{g}(\vec{x}',\tau)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau. \tag{20}$$

Let us examine the terms in the above equation. The term

$$\int_0^t\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}G\tilde{g}(\vec{x}',\tau)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau. \tag{21}$$

denoted by I_g, represents the effect of volume energy generation. The term

$$-\rho c\int_0^t\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}\frac{\partial}{\partial\tau}(GT)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau=-\rho c\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}(GT)|_0^t\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$$
$$=\rho c\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}(GT)\big|_{\tau=0}\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \tag{22}$$

as G vanishes at the upper limit $\tau=t(=t+)$. This term, denoted by I_{in}, represents the effect of the initial condition. We can show the remaining terms in Eq. 20 above represent the effects of the various boundary conditions on the faces $x'=0,L_1$, $y'=0,L_2$, and $z'=0,L_3$. For example, the terms involving x and u become

$$-\int_0^t\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}k_x\left(T\frac{\partial^2G}{\partial x'^2}-G\frac{\partial^2T}{\partial x'^2}\right)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau-\int_0^t\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}\rho cu\frac{\partial}{\partial x'}(GT)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau$$
$$=\int_0^t\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}\left(k_x\frac{\partial}{\partial x'}\left(G\frac{\partial T}{\partial x'}-T\frac{\partial G}{\partial x'}\right)-\frac{\rho cu}{k_x}(GT)\right)\Bigg|_{x'=0}^{x'=L_1}\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau=I_{xL_1}+I_{x0}, \tag{23}$$

where I_{x0} and I_{xL_1} denote, respectively, the effects of the boundary conditions at the faces $x=0$ and $x=L_1$ and are given as

$$I_{x0}=-\int_0^t\int_0^{L_3}\int_0^{L_2}k_x\left(G\frac{\partial T}{\partial x'}-T\frac{\partial G}{\partial x'}-\frac{\rho cu}{k_x}(GT)\right)\Bigg|_{x'=0}\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau, \tag{24}$$

$$I_{xL_1}=\int_0^t\int_0^{L_3}\int_0^{L_2}k_x\left(G\frac{\partial T}{\partial x'}-T\frac{\partial G}{\partial x'}-\frac{\rho cu}{k_x}(GT)\right)\Bigg|_{x'=L_1}\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau. \tag{25}$$

We define the expressions I_{y0}, I_{yL_2}, I_{z0}, and I_{zL_3} similarly and write

$$\rho cT(x,y,z,t)=I_g+I_{\text{in}}+I_{x0}+I_{xL_1}+I_{y0}+I_{yL_2}+I_{z0}+I_{zL_3}. \tag{26}$$

Notice that the above expressions still contain unevaluated integrations in the remaining variables and time. It will be asumed that the boundary conditions are simple enough so that the remaining integrations can be carried out and expressed in closed-form.

4 Boundary conditions for the Green's functions

We now consider issues concerning boundary conditions for the Green's functions. In Eq. 26 various integrals are given that represent the effects of the volumetric energy generation, and the initial and boundary conditions. Let us examine a typical integrand such as that for I_{x0}

$$-k_x\left(G\frac{\partial T}{\partial x'}-T\frac{\partial G}{\partial x'}-\frac{\rho cu}{k_x}GT\right)\Bigg|_{x'=0}. \tag{27}$$

It involves T and its first partial derivative $\partial T/\partial x'$ as well as G and its partial derivative $\partial G/\partial x'$. In order to satisfy boundary condition of the first, the second or the third kind, the integrand above must be made so that the ratio of the coefficient of the $\partial T/\partial x'$ term to that of T is equal to α/β, where α and β refer to those that appear in the general boundary conditions.

$$\alpha\frac{\partial T}{\partial x'}+\beta T=\gamma. \tag{28}$$

This requirement leads to a homogeneous condition that G and its partial derivative must satisfy and makes it possible that the integrand is now uniquely determined with the given boundary condition in terms of the temperature T and its partial derivative.

We consider the three boundary condition cases:

(1) T is prescribed on $x=0$ with $T=T_0$ (boundary condition of the first kind)
We set the coefficient of $\frac{\partial T}{\partial x'}$ in Eq. 27 equal to 0, i.e.,

$$k_xG=0 \tag{29}$$

or simply

$$G=0. \tag{30}$$

With $G=0$ it is seen that the integrand of I_{x0} is uniquely determined with given $T=T_0$ on $x=0$ and

$$I_{x0}=\int_0^t\int_0^{L_3}\int_0^{L_2}k_xT_0\frac{\partial G}{\partial x'}|_{x'=0}\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau. \tag{31}$$

(2) $-\frac{\partial T}{\partial x}=q_{x0}/k_x$ is prescribed on $x=0$ (boundary condition of the second kind)
We set the coefficient of the T term in (27) equal to 0. This leads to

$$k_x\frac{\partial G}{\partial x'}+\rho cuG=0. \tag{32}$$

It follows that

$$I_{x0}=-\int_0^t\int_0^{L_3}\int_0^{L_2}Gq_{x0}|_{x'=0}\mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau. \tag{33}$$

(3) Boundary condition of the third kind
We consider now boundary condition of the type

$$\alpha\frac{\partial T}{\partial x}+\beta T=\gamma \text{ on } x=0. \tag{34}$$

We first determine G so that the ratio of the coefficient of $\frac{\partial T}{\partial x}$ to that of T in the integrand given in Eq. 27 is equal to α/β

$$\frac{\alpha}{\beta}=\frac{-k_xG}{k_x\frac{\partial G}{\partial x'}+\rho cuG} \tag{35}$$

or

$$k_x\alpha\frac{\partial G}{\partial x'}+(\alpha\rho cu+k_x\beta)G=0. \tag{36}$$

This yields a linear, homogeneous expression in G and its normal derivative and simplifies the integrand I_{x0} which is then uniquely determined by the given boundary condition. It can be shown that the integral I_{x0} is now given by

Table 2 Boundary conditions for Green's functions at an x-boundary

B.C. Type	B.C. on T	B.C. on G	B.C. on G^*
1st kind	T is given	$G = 0$	$G^* = 0$
2nd kind	D_xT is given	$k_xD_xG + \rho cuG = 0$	$k_xD_xG^* + \rho cuG^* = 0$
3rd kind	$\alpha D_xT + \beta T$ given	$\alpha k_xD_xG + (\alpha\rho cu + \beta k_x)G = 0$	$\alpha k_xD_xG^* + (\alpha\rho cu + \beta k_x)G^* = 0$

where $D_x$denotes partial differentiation with respect to x. Entries in column 4 are the same as in column 3 except G is replaced by G^*

$$I_{x0} = \int_0^t \int_0^{L_3} \int_0^{L_2} \frac{k_x G\gamma}{\alpha} \mathrm{d}y'\mathrm{d}z'\mathrm{d}\tau. \tag{37}$$

We treat the cases at a y- and a z-boundary similarly, for example, simply letting v or w to replace u and letting y or z to replace x. The results are summarized in Table 2.

5 Construction of the Green's functions

We shall illustrate in this section how to construct the Green's functions. The Green's function G is defined in Eqs. 14 and 15, where it is regarded as a function of x', y', z' and τ. It is known that three-dimensional Cartesian Green's functions may be expressed as products of one-dimensional Green's functions [4]. We thus consider here the one-dimensional Green's function G_X, as defined in Eqs. 14 and 15.

$$k_x \frac{\partial^2 G_X}{\partial x'^2} + \rho c \left(\frac{\partial G_X}{\partial \tau} + u \frac{\partial G_X}{\partial x'} \right) = -\rho c \delta(x - x')\delta(t - \tau),\ t > \tau, \tag{38}$$

For $t > \tau$, the right-hand side above actually vanishes, and G_X is governed by the homogeneous equation,

$$k_x \frac{\partial^2 G_X}{\partial x'^2} + \rho c \left(\frac{\partial G_X}{\partial \tau} + u \frac{\partial G_X}{\partial x'} \right) = 0, \quad t > \tau. \tag{39}$$

We shall characterize the Green's functions as solutions of homogeneous equations that satisfy appropriate initial conditions. We determine the initial conditions of G_X and of G_X^* at $\tau = t$ (or less generally at $\tau = 0$) by integrating the respective nonhomogeneous equation with respect to τ from $\tau = t-$ to $\tau = t+$, then using the fact $G_X \equiv 0$ for $t < \tau$ or $G_X^* \equiv 0$ for $t > \tau$. Now integrating Eq. 38 and keeping only the leading terms, we find

$$\rho c(G_X(x, t; x', t+) - G_X(x, t; x', t-)) = -\rho c \delta(x - x') \int_{t-}^{t+} \delta(t - \tau)\mathrm{d}\tau, \tag{40}$$

which then gives

$$G_X(x, t; x', \tau)|_{\tau = t-} = \delta(x - x'). \tag{41}$$

Here we have our initial time at t. For initial time at $t = 0$ in particular we have

$$G_X(x, 0; x', \tau)|_{\tau = 0-} = \delta(x - x'). \tag{42}$$

Next for G_X^* we obtain from Eqs. 16 and 17

$$k_x \frac{\partial^2 G_X^*}{\partial x'^2} - \rho c \left(\frac{\partial G_X^*}{\partial \tau} + u \frac{\partial G_X^*}{\partial x'} \right) = -\rho c \delta(x - x')\delta(t - \tau). \tag{43}$$

For $t < \tau$ the right-hand side above vanishes and we have

$$k_x \frac{\partial^2 G_X^*}{\partial x'^2} - \rho c \left(\frac{\partial G_X^*}{\partial \tau} + u \frac{\partial G_X^*}{\partial x'} \right) = 0, \quad t < \tau. \tag{44}$$

Now integrating Eq. 43 with respect to τ from $\tau = t-$ to $\tau = t+$, then using the fact $G_X^* \equiv 0$ for $t > \tau$ and keeping only the leading terms yield

$$G^*_X(x,t;x',\tau)|_{\tau=t+} = \delta(x-x') \tag{45}$$

and for $t = 0$

$$G^*_X(x,0;x',\tau)|_{\tau=0+} = \delta(x-x'). \tag{46}$$

The initial conditions given by Eqs. 42 and 46 will be used in the construction of the Green's function G_X and G^*_X.

To find G_X we write it as a product

$$G_X(x,t;x',\tau) = F_X(x')K_X(\tau), \tag{47}$$

where F_X is regarded as a function of x' with the parameter x and K_X is regarded as a function of τ with the parameter t. Substituting Eq. 47 in Eq. 39 and separating the variables leads to the ordinary differential equations

$$k_x F''_X + \rho c u F'_X = -\lambda^2 F_X, \tag{48}$$
$$\rho c K'_X = \lambda^2 K_X, \tag{49}$$

where λ^2 are the spatial eigenvalues and $F = F_X$ are the corresponding eigenfunctions, regarded as functions of x' and subjected to the homogeneous boundary conditions of the kind that G_X satisfies at $x' = 0$ and $x' = L_1$. We note that the appropriate homogeneous boundary conditions for G_X are listed in the third column of Table 2 and the precise entry to be used depends on the type of boundary conditions that T satisfies at $x' = 0$ and $x' = L_1$.

Similarly for the adjoint Green's function G^*_X we write

$$G^*_X(x,t;x',\tau) = H_X(x')\tilde{K}_X(t). \tag{50}$$

Substituting Eq. 50 in Eq. 44 yields

$$k_x H''_X - \rho c u H'_X = -\tilde{\lambda}^2 H_X, \tag{51}$$
$$\rho c \tilde{K}'_X = -\tilde{\lambda}^2 \tilde{K}_X, \tag{52}$$

where $H_X(x')$ satisfies homogeneous boundary conditions at $x' = 0$ and at $x' = L_1$ of the same kind that G^*_X does, which are listed in the fourth column of Table 2, with the precise entry depends on the type of boundary conditions that T satisfies at $x' = 0$ and $x' = L_1$.

For eigenfunctions $F_X(x')$ as functions of x' we seek solutions of the form

$$F_X(x') = \mathrm{e}^{\gamma x'} \tag{53}$$

Substituting in Eq. 48 yields

$$\gamma = \gamma_1, \quad \gamma_2 = \frac{-\rho c u}{2k_x} \pm \frac{\kappa}{L_1}\mathrm{i}, \tag{54}$$

where

$$\kappa = \frac{(4k_x\lambda^2 - (\rho c u)^2)^{1/2}}{2k_x} L_1. \tag{55}$$

We write the solution for $F_X(x')$ as

$$F_X(x') = A_1 P_1(x') + A_2 P_2(x'), \tag{56}$$

where A_1 and A_2 are arbitrary constants and $P_1(x')$ and $P_2(x')$ are the two linearly independent particular solutions of Eq. 48 given by

$$P_1(x') = \exp^{-\frac{\rho c u x'}{2k_x}} \sin(\kappa(x'-L_1)/L_1), \tag{57}$$

$$P_2(x') = \exp^{-\frac{\rho c u x'}{2k_x}} \cos(\kappa(x'-L_1)/L_1). \tag{58}$$

We wish to determine the constants A_1 and A_2 or the ratio thereof such that nontrivial solutions of Eq. 48 exist that satisfy the homogeneous boundary conditions at $x' = 0$ and $x' = L$. We notice that we still have λ as a parameter in the proposed solution for $F_X(x')$ in Eq. 48 above. Applying the two boundary conditions at $x' = 0$ and $x' = L_1$ thus determines the eigenvalue λ and the ratio of A_1 to A_2.

As an example, let us assume that the boundary conditions on T is of the second kind at $x' = 0$ and of the first kind at $x' = L_1$. From Table 2 the corresponding boundary conditions on G, G^* and hence those on F and H are all known.

$$\text{At}\, x' = 0, k_x F_X' + \rho cu F_X = 0, H_X' = 0, \tag{59}$$

$$\text{At}\, x' = L_1, F_X = 0, H_X = 0. \tag{60}$$

We take

$$A_2 = 0, \quad A_1 = 1, \tag{61}$$

so

$$F_X(x') = P_1(x'). \tag{62}$$

It is seen that the chosen $F_X(x')$ already satisfies the boundary conditon at $x' = L_1$. Substituting it in the boundary condition at $x' = 0$ leads to the transcendental equation

$$(\rho cu)\tan\left(\frac{1}{2k_x}(4k_x^2\lambda^2 - (\rho cu)^2)^{1/2}\frac{1}{L_1}\right) = \left(4k_x^2 - (\rho cu)^2\right)^{1/2}, \tag{63}$$

which is the equation that determines the eigenvalues λ^2. Equation 63 may be rewritten as

$$\tan(\kappa) = \kappa/D, \quad D = \frac{\rho cu L_1}{2k_x} = P/2, \tag{64}$$

where $P = 2D$ is the Péclet number in heat transfer.

For the H-problem we proceed similarly and obtain

$$H_{Xn}(x') = \mathrm{e}^{\frac{\rho cu x'}{2k_x}} \sin(\kappa_n(x' - L_1)/L_1). \tag{65}$$

We state that, by applying the boundary conditions on the general solution for H_X at $x' = 0$ and at $x' = L_1$, the same equation for the eigenvalues $\tilde{\lambda}$ as Eq. 64 is obtained. Thus the F-problem and the H-problem have the same set of eigenvalues. The H-eigenfunctions given in Eq. 65 above are, however, different from the F-eigenfunctions. It can be proved that, in general, the eigenvalue problems for $F_X(x')$ and $H_X(x')$, being the adjoint problem to each other, share a same set of eigenvalues, here given by $\tilde{\lambda}_n^2 = \lambda_n^2$, that tend to infinity as n goes to infinity. Furthermore, these eigenfunctions $(F_{Xn}(x'))$ and $(H_{Xn}(x'))$ are "bi-orthogonal", in the sense that

$$\langle F_{Xm}(x'), H_{Xp}(x')\rangle \equiv \int_0^{L_1} F_{Xm}(x')H_{Xp}(x')\mathrm{d}x' = 0, \quad m \neq p \tag{66}$$

and are expected to be "complete" each in the sense

$$\sum_{n=1}^{\infty} \frac{F_{Xn}(x')H_{Xn}(x)}{N_{Xn}} = \delta(x - x'), \tag{67}$$

$$\sum_{n=1}^{\infty} \frac{H_{Xn}(x')F_{Xn}(x)}{N_{Xn}} = \delta(x - x'), \tag{68}$$

which for simplicity will be assumed true here. Proofs of the completeness of eigenfunctions of second-order differential equations can be established using variational methods (see, e.g., B. Friedman, 1956). We note also that N_{Xn} above denotes the "inner product" of F_{Xn} and H_{Xn}

$$N_{Xn} = \langle F_{Xn}(x'), H_{Xn}(x')\rangle. \tag{69}$$

The solutions for time components $K_{Xn}(\tau)$ and $\tilde{K}_{Xn}(\tau)$ are determined from Eqs. 49 and 52 as

$$K_{Xn}(\tau) = C_1(t) \times \mathrm{e}^{\frac{\lambda_n^2 \tau}{\rho c}}, \quad \tau < 0, \tag{70}$$

$$\tilde{K}_{Xn}(\tau) = C_2(t) \times \mathrm{e}^{-\frac{\lambda_n^2 \tau}{\rho c}} \quad \tau > 0, \tag{71}$$

where $C_1(t)$ and $C_2(t)$ are chosen as

$$C_1(t) = \mathrm{e}^{-\frac{\lambda_n^2 t}{\rho c}}, \tag{72}$$

$$C_2(t) = \mathrm{e}^{\frac{\lambda_n^2 t}{\rho c}}, \tag{73}$$

Thus,

$$K_{Xn}(t,\tau) = \mathrm{e}^{-\frac{\lambda_n^2 (t-\tau)}{\rho c}}, \tag{74}$$

$$\tilde{K}_{Xn}(t,\tau) = \mathrm{e}^{-\frac{\lambda_n^2 (\tau-t)}{\rho c}}, \tag{75}$$

which satisfy Eqs. 49 and 52, respectively, when regarded as functions of τ and show the dependence on the parameter.

The Green's functions $G_X(x,t;x,\tau)$ and $G_X^*(x,t;x',\tau)$ are constructed as

$$G_X(x,t;x',\tau) = \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{\lambda_n^2 (t-\tau)}{\rho c}} \frac{F_{Xn}(x') H_{Xn}(x)}{N_{Xn}}, \quad \tau < t, \tag{76}$$

$$G_X^*(x,t;x',\tau) = \sum_{n=1}^{\infty} \mathrm{e}^{\frac{\lambda_n^2 (t-\tau)}{\rho c}} \frac{H_{Xn}(x') F_{Xn}(x)}{N_{Xn}}, \quad \tau > t. \tag{77}$$

We point out that the initial conditions are satisfied at $\tau = t$ for then

$$G_X(x,\tau,x',\tau) = \sum_{n=1}^{\infty} \frac{F_{Xn}(x') H_{Xn}(x)}{N_{Xn}}, \tag{78}$$

$$G_X^*(x,\tau,x',\tau) = \sum_{n=1}^{\infty} \frac{H_{Xn}(x') F_{Xn}(x)}{N_{Xn}}. \tag{79}$$

It is seen that the initial conditions at $t = \tau$ of G_X and of G_X^* are both satisfied by using the completeness properties given by Eqs. 67 and 68.

The one-dimensional Green's functions G_Y, G_Z, G_Y^* and G_Z^* are constructed similarly and the three-dimensional Green's functions G and G^* are formed by the products $G_X G_Y G_Z$ and $G_X^* G_Y^* G_Z^*$, respectively. For example we have

$$G(x,y,z,t;x',y',z',\tau) = \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{(\lambda_n^2 + \mu_m^2 + \nu_p^2)(t-\tau)}{\rho c}} \times \frac{F_{Xn}(x') H_{Xn}(x)}{N_{Xn}} \frac{F_{Ym}(y') H_{Ym}(y)}{N_{Ym}} \frac{F_{Zp}(z') H_{Zp}(z)}{N_{Zp}}, \quad \tau < t, \tag{80}$$

with a similar expression for G^*. It is seen that the expression for G involves the eigenvalues μ_m and ν_p as well as the functions F and H and the norms N_{Ym} and N_{Zp} in the y- and z-directions.

6 Example problems and numerical results

We shall consider two example problems in this section and shall present some related numerical results.

6.1 Problem 1

We consider here a one-dimensional problem on the interval $0 < x < L$. The boundary conditions at both $x = 0$ and $x = L$ are of the first kind, with $T = T_0 \neq 0$ at $x = 0$ and $T = 0$ at $x = L$. This problem is referred to as the XU11B10 problem in the notations of [4].

We are interested in obtaining the steady-state solutions $T_s(x)$ using the Green's function method and compare them with those we shall obtain directly from solving the governing equation for T that is simplified by dropping the terms that depend on time t. Writing $T_s(x)$ for the steady-state solution $T(x,\infty)$, we have for $T_s(x)$

$$k_x \frac{\mathrm{d}^2 T_S(x)}{\mathrm{d}x^2} - \rho c u \frac{\mathrm{d}T_s(x)}{\mathrm{d}x} = 0. \tag{81}$$

Solving Eq. 81 subjected to the boundary conditions at $x = 0$ and $x = L$, we have

$$\frac{T_s(x)}{T_0} = \frac{1 - \mathrm{e}^{-2D(1-\frac{x}{L})}}{1 - \mathrm{e}^{-2D}}. \tag{82}$$

This expression will be used to generate exact solutions.

The Green's functions for the present problem can be written as

$$G(x,t;x',\tau) = \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{R_n^2 \alpha(t-\tau)}{L^2}} \frac{F_{Xn}(x') H_{Xn}(x)}{N_{Xn}}, \tag{83}$$

where

$$F_{Xn}(x') = \mathrm{e}^{-D\frac{x'}{L}} \sin\left(\frac{\kappa_n (x'-L)}{L}\right), \tag{84}$$

$$H_{Xn}(x) = \mathrm{e}^{D\frac{x}{L}} \sin\left(\frac{\kappa_n (x-L)}{L}\right), \tag{85}$$

$$N_{Xn} = \frac{L}{2}, \quad \alpha = k_x/(\rho c), \tag{86}$$

$$R_n^2 = \lambda_n^2 L^2 = \kappa_n^2 + D^2, \quad \kappa_n = (R_n^2 - D^2)^{1/2}. \tag{87}$$

We note that with the functions F_{Xn} and H_{Xn} chosen above the boundary conditions at $x = L$ ($x' = L$) are already satisfied for all κ. The boundary conditions at $x = 0$ ($x' = 0$) are satisfied when

$$\sin(\kappa) = 0. \tag{88}$$

Thus

$$\kappa = n\pi \tag{89}$$

and all $\kappa's$ are determined and no numerical work is necessary.

The solution for the temperature $T(x,t)$ is represented in terms of the Green's function and the boundary data at $x = 0$ as given in Sect. 3 is

$$T(x,t) = I_{X0} = \int_0^t k_x T_0 \frac{\partial G}{\partial x'}|_{x'=0} \mathrm{d}\tau \tag{90}$$

We differentiate G in Eq. 83 with respect to x' and then evaluate it at $-x'$ (outward pointing normal) $= 0$

$$-\frac{\partial G(x,t;0,\tau)}{\partial x'} = \frac{2\pi}{L^2} \mathrm{e}^{Dx/L} \sum_{n=1}^{\infty} \mathrm{e}^{-R_n^2 \frac{\alpha(t-\tau)}{L^2}} n \sin(n\pi x/L). \tag{91}$$

Using Eq. 91 in Eq. 90, we find the solution for the temperature $T(x,t)$

$$T(x,t) = 2T_0 \mathrm{e}^{Dx/L} \sum_{n=1}^{\infty} \left(1 - \mathrm{e}^{-R_n^2 \frac{\alpha t}{L^2}}\right) \frac{(n\pi)\sin(n\pi x/L)}{R_n^2}. \tag{92}$$

Table 3 Calculations for the steady state temperature $T_s(x)$ for the $XU11B10$ case. Six terms in the infinite series are used

No. of terms	x/L	$D=-1$		D=0.005		$D=1$	
		T series	T exact	T series	T exact	T series	T exact
6	0.00	0.000	1.000	0.000	1.000	0.000	1.000
6	0.01	0.116	0.977	0.120	0.990	0.118	0.997
6	0.25	0.523	0.545	0.723	0.751	0.863	0.898
6	0.50	0.300	0.269	0.553	0.501	0.816	0.731
6	0.75	0.124	0.102	0.299	0.251	0.556	0.455
6	1.00	0.000	0.000	0.000	0.000	0.000	0.000

Table 4 Calculations for the steady state temperature $T_s(x)$ for the $XU11B10$ case. 250 terms in the infinite series are used

No. of terms	x/L	$D=-1$		D=0.005		$D=1$	
		T series	T exact	T series	T exact	T series	T exact
250	0.00	0.000000	1.000000	0.000000	1.000000	0.000000	1.000000
250	0.01	0.968927	0.977099	0.981794	0.990049	0.988500	0.996838
250	0.25	0.545924	0.544946	0.752194	0.750937	0.900076	0.898464
250	0.50	0.269714	0.268941	0.502526	0.501250	0.733158	0.731059
250	0.75	0.100936	0.101536	0.249663	0.250938	0.452365	0.455054
250	1.00	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

The steady-state portion of this solution is

$$T_s(x) = 2T_0 \mathrm{e}^{Dx/L} \sum_{n=1}^{\infty} \frac{(n\pi)\sin(n\pi x/L)}{R_n^2}. \tag{93}$$

Calculations for the steady state temperature based on six-term truncations of the infinite series are given in Table 3 for $D=-1, 0.01$, and 1 and for x/L values of 0, 0.01, 0.25, 0.5, 0.75 and 1, 0, and 1. The exact solutions based on Eq. 82 are also shown. Similar calculations for infinite series truncated after 250 terms are given in Table 4.

In the Green's function method presented here, solutions for the temperatures are expressed as integrals involving the Green's functions and prescribed data. Approximations to the solutions are obtained when the infinite series are truncated after retaining just a finite number of terms. Convergence concerns the issue that, when more and more terms in the infinite series are included in the approximations, whether the errors become smaller and smaller.

The steady-state solution $T_s(x)$ represented by Eq. 98 converges well for large n and for all $x \neq 0$ or L, i.e., away from the boundary. The solution for T_s at $x=L$ $(T_s=0)$ is, in fact, exact. Convergence is poor as $x=0$ is approached with many oscillations. Also, the solution given by Eq. 93 at $x=0$ appears to be zero, which contradicts the fact that T_S is prescribed at $x=0$ to be T_0 which need not be zero.

The above contradiction may be explained by resorting to the theory of Fourier series. The solution T_s given here as a function of x is in fact discontinuous at $x=0$ and the limit of T_s as x tends to 0, (which by definition is the boundary condition at $x=0$), is not equal to the value of T_s evaluated at $x=0$. More specifically we can show that by writing $T_s(x)$ as

$$T_s(x) = 2T_0 \mathrm{e}^{Dx/L} S(x). \tag{94}$$

The infinite series $S(x)$ has a "dominant" part $S_1(x)$(in the sense that $S-S_1$ as a function of x converges uniformly and hence to an everywhere continuous function of x. This $S_1(x)$, given by

$$S_1(x) = \sum_{n=1}^{\infty} \frac{\sin(n\pi x/L)}{n\pi}, \tag{95}$$

converges slowly, but has a closed-form sum given by

$$S_1(x) = \frac{L - x}{2L}, \quad 0 < x < L. \tag{96}$$

Thus, as x goes to zero, S_1 as a function of x goes to 1/2 and T_s as a function of x goes to T_0.

6.2 Problem 2

We consider now a three-dimensional heat-conduction problem with solid flow. The solid velocity is assumed in the x-direction, $\vec{u} = (u, 0, 0)$. The solid is taken to be a cube with side length L and $0 < x < L, 0 < y < L$ and $0 < z < L$. The boundary conditions are

$$\begin{aligned}
&\text{On } x = 0,\ k_x \frac{\partial T}{\partial x} = -q_0 = \text{constant},\quad \text{On } x = L,\ T = 0,\\
&\text{On } y = 0,\ T = 0,\ \text{On } y = L,\ \frac{\partial T}{\partial y} = 0,\\
&\text{On } z = 0,\ T = 0,\ \text{On } z = L,\ \frac{\partial T}{\partial z} = 0.
\end{aligned} \tag{97}$$

Also we have the initial condition

$$T(x, y, z, 0) = 0. \tag{98}$$

We shall assume, for simplicity, that physical parameters such as k, ρ, c and α, etc, are the same in the x-, y-, and z-direction. Furthermore, we shall assume that $D \ngtr 1$ so that the term involving hyperbolic functions is absent from the series for the Green's functions. This problem is known as the XU21B10Y12B00Z12B00T0 problem in the notations of [4].

The three-dimensional Green's function G is written as a product of the one-dimensional Green's functions,

$$G(x, x', y, y'z, z', \sigma) = G_X(x, x', \sigma) G_Y(y, y', \sigma) G_Z(z, z', \sigma), \tag{99}$$

where use has been made of the fact that the dependence of the Green's functions on t and τ is only through some $\sigma = t - \tau$. We shall refer to σ as "cotime".

We have

$$G_X(x, x', \sigma) = \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n^2 \alpha\sigma} \frac{F_{Xn}(x') H_{Xn}(x)}{N_{Xn}}, \tag{100}$$

$$G_Y(y, y', \sigma) = \sum_{m=1}^{\infty} \mathrm{e}^{-\mu_m^2 \alpha\sigma} \frac{F_{Ym}(y') H_{Ym}(y)}{N_{Ym}}, \tag{101}$$

$$G_Z(z, z', \sigma) = \sum_{p=1}^{\infty} \mathrm{e}^{-\nu_p^2 \alpha\sigma} \frac{F_{Zp}(z') H_{Zp}(z)}{N_{Zp}}, \tag{102}$$

where

$$F_{Xn}(x') = \mathrm{e}^{-\frac{\rho c u x'}{2k_x}} \sin(\kappa(x' - L)/L), \tag{103}$$

$$H_{Xn}(x) = \mathrm{e}^{\frac{\rho c u x}{2k_x}} \sin(\kappa(x - L)/L), \tag{104}$$

$$F_{Ym}(y') = \sin(\beta y'/L),\quad H_{Ym}(y) = \sin(\beta y/L), \tag{105}$$

$$\beta_m = (2m - 1)(\pi/2), \tag{106}$$

$$F_{Zp}(z') = \sin(\eta z'/L), H_{Zp}(z) = \sin(\eta z/L), \tag{107}$$

$$\eta_p = (2p - 1)(\pi/2). \tag{108}$$

We recall that in the x-direction, λ^2 are related to κ^2 by Eq. 55. This fact enables us to write the Green's function G_X as

$$G_X(x,x',\sigma) = \frac{2}{L}\mathrm{e}^{\frac{Dx}{L}} \sum_{n=1}^{\infty} \mathrm{e}^{-R_n^2\frac{\alpha\sigma}{L^2}} \times \frac{R_n^2 \sin(\kappa\frac{L-x}{L})\sin(\kappa_n)}{R_n^2 - D}, \tag{109}$$

where

$$R_n^2 = \lambda_n^2 k/L^2 = \kappa^2 + D^2, \quad \tan(\kappa_n) = \kappa_n/D, \quad D = \frac{uL}{2\alpha}. \tag{110}$$

We mention that the roots κ of Eq. 64 are all real for $D < 1.0$. Hence κ^2 are all real and positive and they may be arranged as an increasing sequence of n. For $D = 1$ the first root (the smallest in absolute value) for κ_1 is zero and there is no non-trivial eigenfunction. For $D > 1$, there exist a pair of purely imaginary roots for κ, leading to a real negative eigenvalue κ^2. The corresponding spatial eigenfunctions F and H are now in the form of hyperbolic functions instead of the trigonometric functions. The quantities λ^2, however, remain real and positive, consistent with the time-decaying nature of the solutions to steady state solutions for large t. We shall assume that $D \not> 1$ so that this extraneous root for κ associated with the hyperbolic eigenfunction does not arise. For details, please see [6, 7].

In the y-direction, μ^2 are related to γ^2 by

$$\gamma^2 = \frac{\mu^2 L^2}{k}. \tag{111}$$

This enables us to write $G_Y(y,y',\sigma)$ as

$$G_Y(y,y',\sigma) = \frac{2}{L}\sum_{m=1}^{\infty} \mathrm{e}^{-\gamma_m^2\frac{\alpha\sigma}{L^2}} \sin\left(\gamma_m\frac{y}{L}\right)\sin\left(\gamma_m\frac{y'}{L}\right). \tag{112}$$

According to Sect. 3 the temperature $T(x,y,z,t)$ is given by

$$T(x,y,z,t) = \frac{\alpha q_0}{k}\int_{\sigma=0}^{t} G_X(x,0,\sigma)\int_{y'=0}^{L} G_y(y,y',\sigma)\mathrm{d}y' \int_{z'=0}^{L} G_Z(z,z',\sigma)\mathrm{d}z'\mathrm{d}y'\mathrm{d}\sigma. \tag{113}$$

We carry out the y'- and z'-integration and obtain

$$\begin{aligned}\int_{y'=0}^{L} G_Y(y,y',\sigma)\mathrm{d}y' &= \int_{y'=0}^{L}\frac{2}{L}\sum_{m=1}^{\infty} \mathrm{e}^{-\gamma_m^2\frac{\alpha\sigma}{L^2}} \sin\left(\gamma_m\frac{y}{L}\right)\sin\left(\gamma_m\frac{y'}{L}\right)\mathrm{d}y' \\ &= 2\sum_{m=1}^{\infty} \mathrm{e}^{-\gamma_m^2\frac{\alpha\sigma}{L^2}} \sin\left(\gamma_m\frac{y'}{L}\right)/\gamma_m\end{aligned} \tag{114}$$

plus a similar expression from the z'-integration. Next we obtain, upon performing the σ-integration in Eq. 114

$$T(x,y,z,t) = -T_{\mathrm{c.t.}}(x,y,z,0) + T_{\mathrm{c.t.}}(x,y,z,t), \tag{115}$$

where

$$\frac{T_{\mathrm{c.t.}}(x,y,z,t)}{q_0 L/k} = -8\mathrm{e}^{\frac{Dx}{L}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \mathrm{e}^{-(R_n^2+\gamma_m^2+\eta_p^2)\frac{\alpha t}{L^2}} \times \frac{R_n^2\sin(\kappa_n\frac{L-x}{L})\sin(\kappa_n)\sin(\gamma_m\frac{y}{L})\sin(\eta_p\frac{z}{L})}{(R_n^2-D)(R_n^2+\gamma_m^2+\eta_p^2)\gamma_m\eta_p}. \tag{116}$$

The last equation has a c.t. subscript which we use to denote "complementary transient".

The complementary transient converges exponentially for all time except for $t = 0$. At time zero, it gives the steady-state solution. The first term on the right of Eq. 115 is the steady-state term. A direct evaluation of the steady-state term from Eq. 116 shows slow convergence, particularly near $x = 0$. Several means are

Table 5 Numerical results for the complementary transient for a cube are presented

D	x/L	y/L	z/L	Dim. t	Total no. of terms	$T_{\text{c.t.}}$
−0.50	0.50	1.00	1.00	0.0020	20838	−0.1187766466
−0.50	0.50	1.00	1.00	0.0030	11273	−0.1187766466
−0.50	0.50	1.00	1.00	0.0120	1389	−0.1187371733
−0.50	0.50	1.00	1.00	0.1000	52	−0.0789037698
−0.50	0.50	1.00	1.00	0.3000	9	−0.0152339528
−0.50	0.50	1.00	1.00	1.0000	2	−0.0000381705
0.50	0.50	1.00	1.00	0.0020	20838	−0.3088397305
0.50	0.50	1.00	1.00	0.0030	11321	−0.3088397305
0.50	0.50	1.00	1.00	0.0120	1389	−0.3087720469
0.50	0.50	1.00	1.00	0.1000	52	−0.2290562314
0.50	0.50	1.00	1.00	0.3000	9	−0.0655029313
0.50	0.50	1.00	1.00	1.0000	2	−0.0006722472

Solution with 10 place accuracy are obtained for the location $x/L = 0.5, y/L = z/L = 1$ and for $D = -0.5$ and 0.5.

Table 6 Effects of varying the number of terms in the series of the complementary transient evaluated at $t = 0$ for $D = -0.5, x/L = 0.5, y/L = z/L = 1$

D	x/L	y/L	z/L	t	$m_{\max}$	$n_{\max}$	$p_{\max}$	$T_{\text{c.t.}}$
−0.50	0.50	1.00	1.00	0.0000	25	15	15	−0.1189600716
−0.50	0.50	1.00	1.00	0.0000	25	25	25	−0.1189567530
−0.50	0.50	1.00	1.00	0.0000	100	25	25	−0.1187765234
−0.50	0.50	1.00	1.00	0.0000	200	25	25	−0.1187766312
−0.50	0.50	1.00	1.00	0.0000	200	50	50	−0.1187766317
−0.50	0.50	1.00	1.00	0.0000	250	50	50	−0.1187766542

available to improve the convergence. One is to use time-partitioning [8]. This uses the short cotime Green's functions, as well as the long cotime forms given above. Another approach is to use a known temperature on the left-hand side of Eq. 116. Two types of temperatures are known. One is for location away from the heated surface ($x = 0$ here). The second is for the heated surface or near it but away from $y = 0$ or $z = 0$; at these locations and at sufficiently small dimensionless times the temperature is one-dimensional which is known.

Table 5 displays some results for the complementary transient solution given by Eq. 116 for the location of $x/L = 0.5, y/L = z/L = 1$ and two different values of D. Ten digit accuracy is given. Notice that as the dimensionless time is decreased that the values go to a contant, which is actually the steady state. If the dimensionless time is made smaller, then more terms are needed.

Table 6 shows the effect of varying the number of terms in the series for the complemetary transient evaluated at $t = 0$ for $D = 0.5, x/L = 0.5, y/L = z; /L = 1$.

7 Discussions and concluding remarks

We have studied in this paper heat conduction in a rectangular parallelepiped that is in steady motion relative to a fluid. The solid is assumed orthotropic with (unequal) thermal conductivities k_x, k_y, and k_z. It is also assumed that the principal directions of the thermal conductivities coincide with the coordinate axes. It is seen that no extra effort is required to treat this orthotropic case than the isotropic case. We mention also that there exists a coordinate transformation that transforms the governing equation for the orthotropic case to one for an isotropic case with an equivalent thermal conductivity $k = (k_x k_y k_z)^{1/3}$.

It is seen that the consideration of the solid motion introduces lower-order derivative terms in the governing equation and causes the spatial part of the equation to be non-self-adjoint. The classical heat equation must be modified to accommodate this change. We find it necessary to consider both the Green's function G and its adjoint function G^*. Also, G and G^* satisfy in general different but "commitant" boundary conditions which are studied here as they are needed for establish certain theoretical properties such as the bi-orthogonality of the sets of the spatial eigenfunctions $F's$ and $H's$.

As we have indicated earlier, solutions for the temperature problems here are expected to converge rapidly in the interior of the region. Near a boundary point we have a different story. Solutions near a boundary point may fail to converge, as we have seen in example Problem 1 above near $x = 0$. This is a Fourier series phenomenon. The linear function with $T = T_0$ at $x = 0$ and $T = 0$ at $x = L$ is extended as an odd, $2L$-periodic function of x outside of the interval $0 < x < L$ with a discontinuity at $x = 0$, i.e., a jump in T from $-T_0$ to T_0 as x crosses $x = 0$. The full Fourier series at $x = 0$ converges slowly but ultimately to the average of the two end values L_0 and $-L_0$ which is zero while away from $x = 0$ the Fourier series converge to the value of the function about which the Fourier series is developed. The nonconvergence of the Fourier series at $x = 0$ and the associated oscillations are just a part of the well-known Gibbs' phenomenon [1, p. 745].

We already mentioned that the method of time-partitioning can be used to improve the convergence of the solution of Problem 2 above. This is actually a more general method. In this method, the τ integration is rewritten as an integration with respect to the dummy variable $\sigma = t - \tau$. For small values of σ, which we refer to as a "cotime" we use the small time Green's functions, for instance, Green's functions of the Laplace transform type and for large values of σ we use the large time Green's functions of the kind obtained by the method of separation of variables. See the recent work by the present authors and others in [8].

We mention finally that, as we indicated in [4], it is known that there exists a coordinate transformation that transforms a solution $T(\vec{x}, t)$ of the governing equation here with solid flow to a solution $W(\vec{x}, t)$ of the standard heat equation. Thus the problem that we have treated here could be obtained by: (i) transform the govening equations along with the initial and boundary conditions from the original solution space to the transformed space: (ii) solve the initial-boundary-value problem for the classical heat equation in the transformed space using the transformed data; and (iii) transform the solution obtained in the transformed space back to the original solution space. This is no easy task, especially in the coordinate transformation to and from the solution space. In our work here we avoided the two transformations by staying with the original solution space, thus showing the existence of this transformation is not essential for the solution of this problem. The price that we paid here is that we now have to solve a more complicated set of equations. One of us has in fact attempted to follow the former approach to solve some simple boundary-value problems with solid flow [7, 9]. In view of the fact that so little has been done on this subject, however, it seems premature to draw conclusion as to which method is the more superior at this stage.

Appendix A The reciprocity property of G and G*

We shall derive in this Appendix the so-called reciprocity property of the Green's functions G and G^* given in Eqs. 12 and 13.

We rewrite (7) for $G(\vec{x}, t; \vec{x}_0, t_0)$ where it is assumed that $t > t_0$ and multiply it with $G^*(\vec{x}, t; \vec{x}_1, t_1)$ where it is assumed that $t < t_1$

$$k_x G^* \frac{\partial^2 G}{\partial x^2} + k_y G^* \frac{\partial^2 G}{\partial y^2} + k_z G^* \frac{\partial^2 G}{\partial z^2} - \rho c \left(G^* \frac{\partial G}{\partial t} + u G^* \frac{\partial G}{\partial x} + v G^* \frac{\partial G}{\partial y} + w G^* \frac{\partial G}{\partial z} \right)$$
$$= -\rho c G^* \delta(\vec{x} - \vec{x}') \delta(t - \tau), \ t > \tau. \tag{117}$$

Next we rewrite Eq. 10 for $G^*(\vec{x}, t; \vec{x}_1, t_1)$ and multiply it with $G(\vec{x}, t; \vec{x}_0, t_0)$

$$k_x G \frac{\partial^2 G^*}{\partial x^2} + k_y G \frac{\partial^2 G^*}{\partial y^2} + k_z G \frac{\partial^2 G^*}{\partial z^2} + \rho c \left(G \frac{\partial G^*}{\partial t} + u G \frac{\partial G^*}{\partial x} + v G \frac{\partial G^*}{\partial y} + w G \frac{\partial G^*}{\partial z} \right)$$
$$= -\rho c G \delta(\vec{x} - \vec{x}') (t - \tau), \ t < \tau. \tag{118}$$

We subtract these two equations and integrate the resulting equation with respect to t from $-\infty$ to t_1^+, with respect to z from 0 to L_3, with respect to y from 0 to L_2, and with respect to x from 0 to L_1. We thus have

$$\begin{aligned}
II(G,G^*) \equiv & \int_{-\infty}^{t_1^+}\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}\left(k_x\left(G^*\frac{\partial^2 G}{\partial x^2}-G\frac{\partial^2 G^*}{\partial x^2}\right)\right.\\
& \left.+k_y\left(G^*\frac{\partial^2 G}{\partial y^2}-G\frac{\partial^2 G^*}{\partial y^2}\right)+k_z\left(G^*\frac{\partial^2 G}{\partial z^2}-G\frac{\partial^2 G^*}{\partial z^2}\right)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\
& -\rho c\int_{-\infty}^{t_1^+}\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}\left(\left(G^*\frac{\partial G}{\partial t}+G\frac{\partial G^*}{\partial t}\right)+u\left(G^*\frac{\partial G}{\partial x}+G\frac{\partial G^*}{\partial x}\right)\right.\\
& \left.+v\left(G^*\frac{\partial G}{\partial y}+G\frac{\partial G^*}{\partial y}\right)+w\left(G^*\frac{\partial G}{\partial z}+G\frac{\partial G^*}{\partial z}\right)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\
= & -\rho c\int_{-\infty}^{t_1^+}\int_0^{L_3}\int_0^{L_2}\int_0^{L_1} G^*(\vec{x},t;\vec{x}_1,t_1)\delta(\vec{x}-\vec{x}_0)\delta(t-t_0)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\
& +\rho c\int_{-\infty}^{t_1^+}\int_0^{L_3}\int_0^{L_2}\int_0^{L_1} G(\vec{x},t;\vec{x}_0,t_0)\delta(\vec{x}-\vec{x}_1)\delta(t-t_1)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t. \qquad (119)
\end{aligned}$$

The t-integration of the term involving the t-derivatives of G and G^* on the left of Eq. 119 can be carried out to yield

$$\begin{aligned}
& -\rho c\int_{-\infty}^{t_1^+}\int_0^{L_3}\int_0^{L_2}\int_0^{L_1}\left(G^*\frac{\partial G}{\partial t}+G\frac{\partial G^*}{\partial t}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t\\
& = -\rho c\int_0^{L_3}\int_0^{L_2}\int_0^{L_1} G^*(\vec{x},t;\vec{x}_1,t_1)G(\vec{x},t;\vec{x}_0,t_0)|_{-\infty}^{t_1}\mathrm{d}x\mathrm{d}y\mathrm{d}z. \qquad (120)
\end{aligned}$$

The integrand above vanishes at the lower limit of intgration because G does and vanishes at the upper limit because G^* does.

The remaining terms in $II(G,G^*)$ on the left-hand side of Eq. 119, which is a bilinear form in G and G^* as defined, all vanish. This is due to the fact that at the spatial boundary both G and G^* satisfy the same homogeneous boundary conditions and cancellations occur. Details of the arguments will be omitted here.

Thus, from the right-hand side of Eq. 120 above we obtain

$$G^*(\vec{x}_0,t_0;\vec{x}_1,t_1) = G(\vec{x}_1,t_1;\vec{x}_0,t_0), \qquad (121)$$

which leads to Eq. 13. Equation 12 follows by using the definition of the adjoint function G^*.

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